Single-Agent Reinforcement Learning

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We are here

Outline

Cornerstones

- Markov Decision Process (MDP)
- Bellman Equation

Value-based RL - Model-based Learning

Dynamic Programming

- Policy Iteration
- Value Iteration

Value-based RL – Model-Free Learning

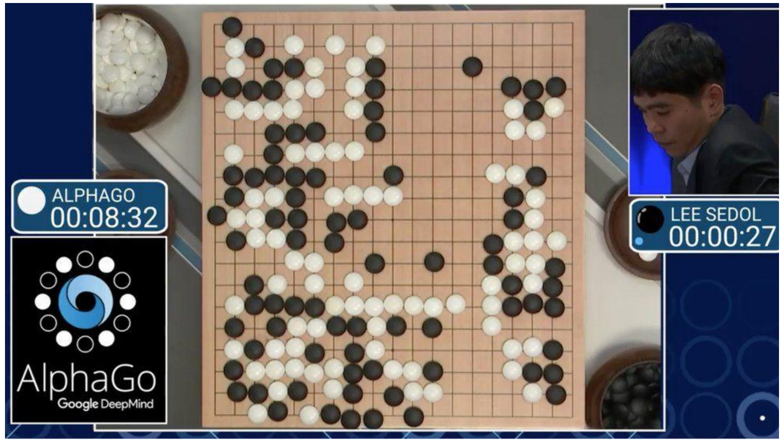
Temporal-Difference Learning

- SARSA
- Q-Learning

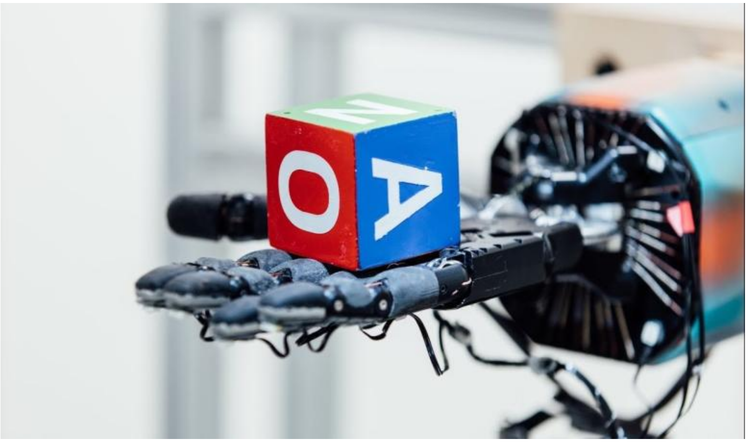
Policy-based RL

- Policy Gradient

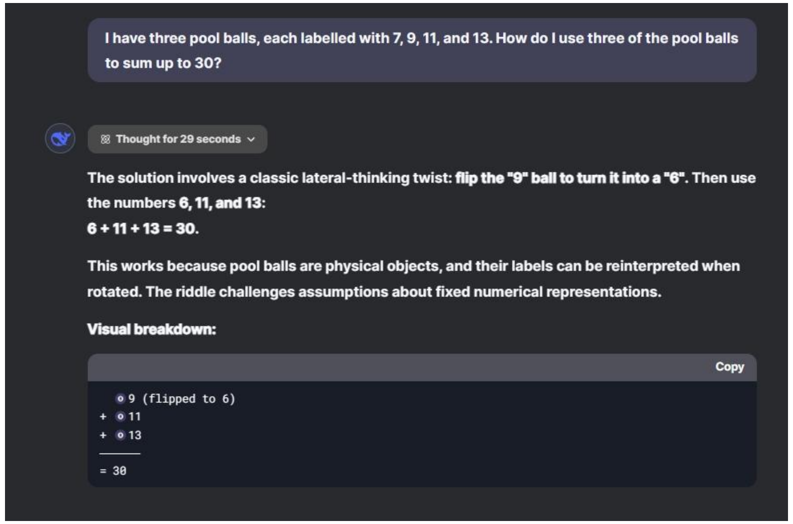
Reinforcement Learning (RL)



AlphaGo in playing Go (MCTS+RL)

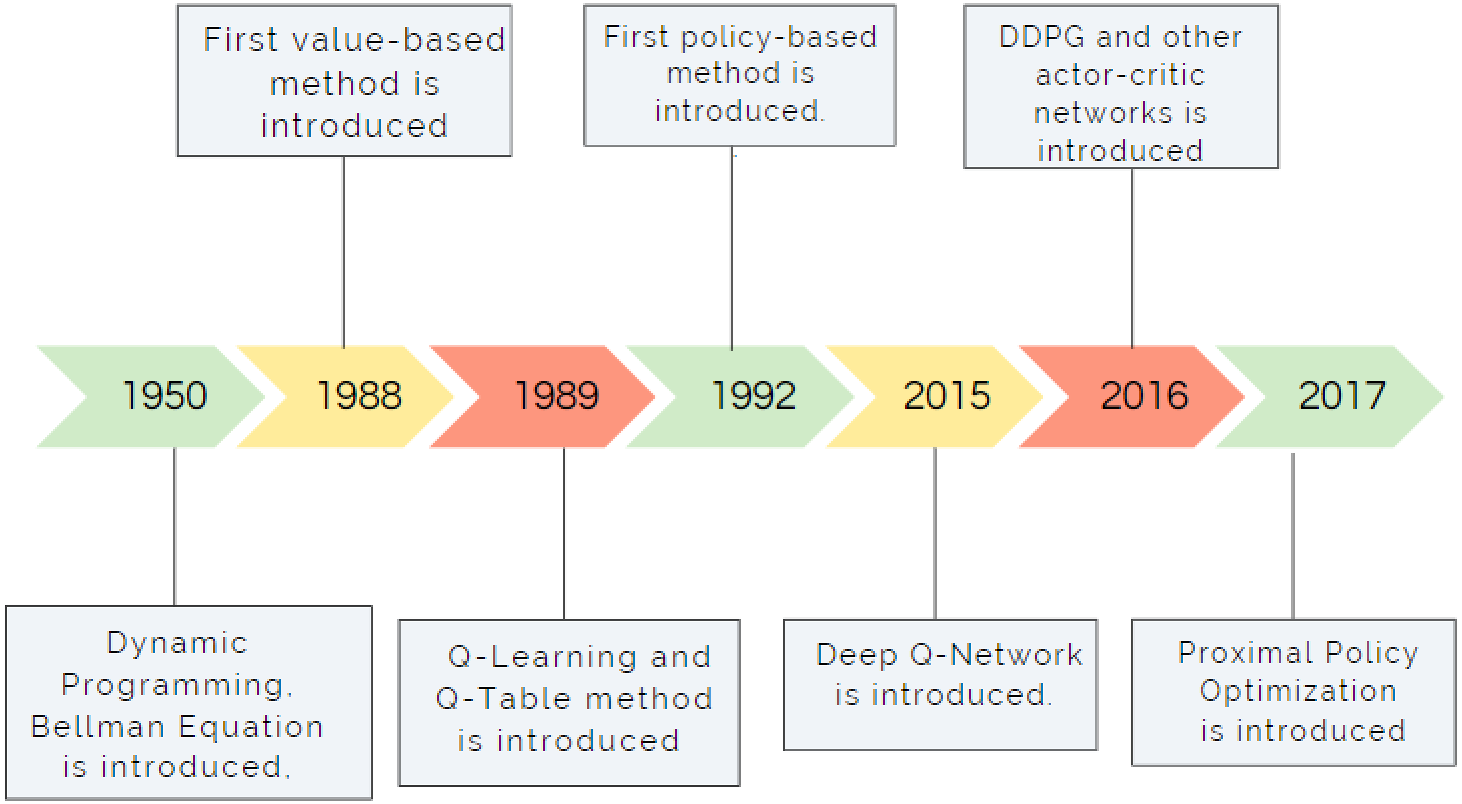


OpenAI's Dactyl in solving Rubik's Cube with a robot hand (Robotic+RL)



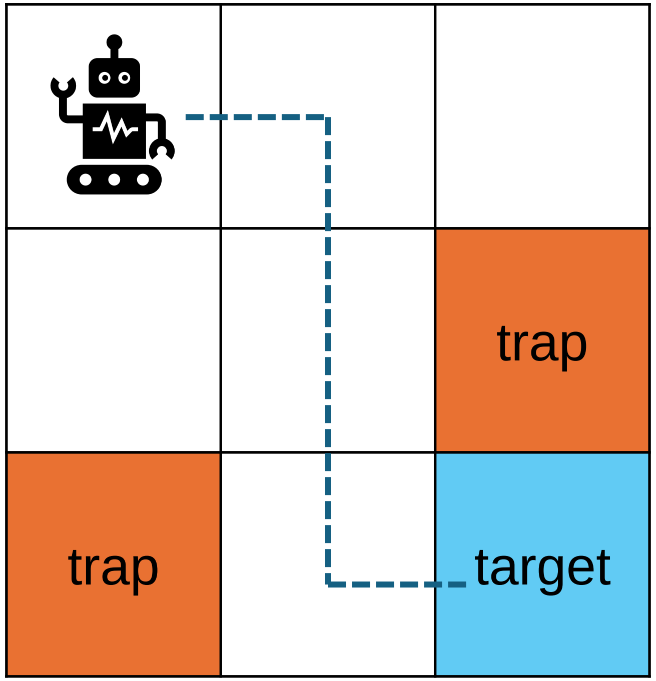
DeepSeek R1 in solving math problems (LLM+RL)

History of Reinforcement Learning



https://medium.com/@sanghaviharsh666/navigating-the-evolution-of-reinforcement-learning-a-historical-perspective-6fbbaa010351

Markov Decision Process: A Deterministic Example

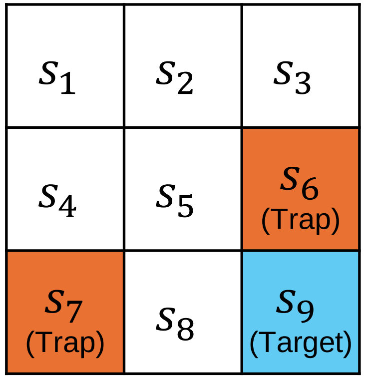


- A robot moves in a grid world
- The robot (called agent) can move across adjacent cells in the grid.
- Target: find a “good” policy that enables it to reach the target cell, when starting from any initial cell. The agent should reach the target without
  - entering any trap cells
  - taking unnecessary detours
  - colliding with the boundary of the grid

Zhao, Shiyu. *Mathematical Foundations of Reinforcement Learning*. Singapore: Springer Nature Singapore, 2025. https://doi.org/10.1007/978-981-97-3944-8.

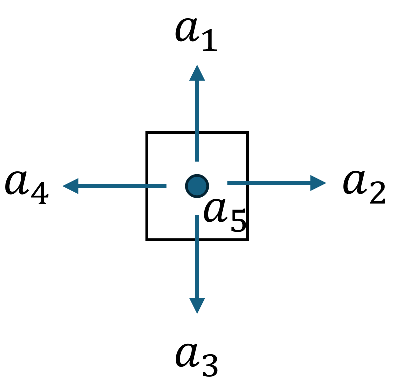
Markov Decision Process: State, Action and State Transition

State



9 cells → 9 states
State space: all states in the problem, $S = \{s_1, \dots, s_9\}$

Action



Move up, right, down, left and stay still
Action space: all possible actions in the problem, $A = \{a_1, \dots, a_5\}$

State Transition

Given a state s and an action a, what will be the next state $P(s, a)$?
e.g., $P(s_1, a_1) = s_2$, $P(s_5, a_3) = s_8$

State \ Action	a_1 (↑)	a_2 (→)	a_3 (↓)	a_4 (←)	a_5 (o)
s_1	s_1	s_2	s_4	s_1	s_1
s_2	…	…	…	…	…
s_3	…	…	…	…	…
s_4	…	…	…	…	…
s_5	…	…	…	…	…
s_6	…	…	…	…	…
s_7	…	…	…	…	…
s_8	…	…	…	…	…
s_9	s_6	s_9	s_9	s_8	s_9

Markov Decision Process: Reward and Agent's Policy

s_1	s_2	s_3
s_4	s_5	s_6 (Trap)
s_7 (Trap)	s_8	s_9 (Target)

- If the agent attempts to exit the boundary, let r = −1.
- If the agent attempts to enter a forbidden cell, let r = −1.
- If the agent reaches the target state, let r = +1.
- Otherwise, the agent obtains a reward of r = 0.

Reward

After executing an action a at a state s, the agent obtains a reward, denoted as $r(s, a)$, as feedback from the environment.

State \ Action	a_1 (↑)	a_2 (→)	a_3 (↓)	a_4 (←)	a_5 (o)
s_1	-1	0	0	-1	0
s_2	-1	0	0	0	0
s_3	-1	-1	-1	0	0
s_4	0	0	-1	-1	0
s_5	0	-1	0	0	0
s_6	0	-1	+1	0	-1
s_7	0	0	-1	-1	-1
s_8	0	+1	-1	-1	0
s_9	-1	-1	-1	0	+1

Agent's Policy

Which actions to take at every state s. E.g., the robot can take two different policies $\pi_1(s)$ and $\pi_2(s)$ are as follows

State	$\pi_1(s)$	$\pi_2(s)$
s_1	→	↓
s_2	↓	↓
s_3	←	↓
s_4	→	↓
s_5	↓	↓
s_6	↓	↓
s_7	→	→
s_8	→	→
s_9	o	o

π_1

→	↓	←
→	↓	↓
→	→	o

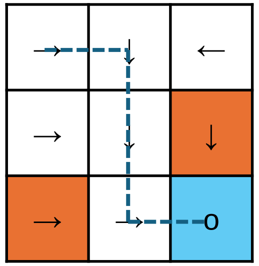
π_2

↓	↓	↓
↓	↓	↓
→	→	o

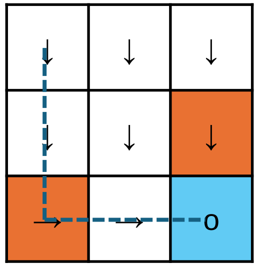
Markov Decision Process: Trajectory and Return

Trajectory

π_1



π_2



Given an initial state s_{init} and the agent's policy π, the agent can start from s_{init} and recurrently follow π to take actions.

$s_0 = s_{init}$
For $t = 0,1, \ldots$
- take an action $a_t \leftarrow \pi(s_t)$
- obtain a reward $R_t \leftarrow r(s_t, a_t)$
- transit to a new state $s_{t+1} \leftarrow P(s_t, a_t)$

For example:

π_1, start from s_1: $s_1 \xrightarrow[r=0]{a_2} s_2 \xrightarrow[r=0]{a_3} s_5 \xrightarrow[r=0]{a_3} s_8 \xrightarrow[r=1]{a_2} s_9$

π_2, start from s_1: $s_1 \xrightarrow[r=0]{a_3} s_2 \xrightarrow[r=-1]{a_3} s_5 \xrightarrow[r=0]{a_3} s_8 \xrightarrow[r=1]{a_2} s_9$

Return (value) of a trajectory

Given an initial state s and the agent's policy π, the return (value) $V_\pi(s)$ is the sum of all rewards on the trajectory

E.g., for the two trajectories, their return is

$$V_{\pi_1}(s_1) = 0 + 0 + 0 + 1 = 1$$
$$V_{\pi_2}(s_1) = 0 - 1 + 0 + 1 = 0$$

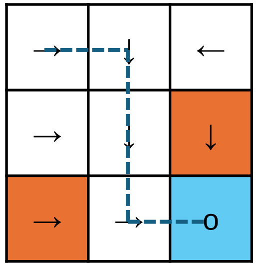
We would like to find a policy for the agent that maximize its return. Therefore, return can be used to evaluate policies.
- For the example, $V_{\pi_1}(s_1) > V_{\pi_2}(s_1)$ so policy π_1 is better than policy π_2

Policy Evaluation and Bellman Equation

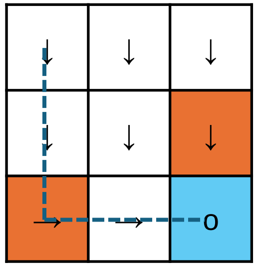
Given the agent's policy π, policy evaluation is to compute $V_\pi(s)$ for all $s \in S$

For a given π, this can be done by listing the trajectory for each state and add up the rewards

π_1



π_2



State	$V_{\pi_1}(s)$	$V_{\pi_2}(s)$
s_1	1	0
s_2	1	1
s_3	1	0
s_4	1	0
s_5	1	1
s_6	1	1
s_7	1	1
s_8	1	1
s_9	0	0

However, we have a more general way to evaluate the policy with **Bellman equation**.

Note that each return consists of an immediate reward and future rewards.

$$\overbrace{\underbrace{s \xrightarrow[R=r(s,a)]{a=\pi(s)}}_{r(s,\pi(s))} \underbrace{s' \xrightarrow[R'=r(s',a')]{a'=\pi(s')} s'' \xrightarrow[R''=r(s'',a'')]{a''=\pi(s'')} \cdots}_{V_\pi(s')}}^{V_\pi(s)}$$

- **Immediate reward**: the reward obtained after taking an action $a = \pi(s)$. That is, $r(s, \pi(s))$
- **Future rewards**: the return of the trajectory starting at the next state $s' = P(s, \pi(s))$

Therefore,

$$V_\pi(s) = r(s, \pi(s)) + V_\pi(s')$$

Which is the Bellman equation

Policy Evaluation with Bellman Equation

π_2

↓	↓	↓
↓	↓	↓
→	→	o

Given the Bellman equation

$$V_\pi(s) = \underbrace{r(s,\pi(s))}_{\text{Immediate reward}} + \underbrace{V_\pi(s')}_{\text{return starting from next state}}$$

We take π shown on the left as an example:

$$V_\pi(s_1) = 0 + V_\pi(s_4)$$
$$V_\pi(s_2) = 0 + V_\pi(s_5)$$
$$V_\pi(s_3) = -1 + V_\pi(s_6)$$
$$V_\pi(s_4) = -1 + V_\pi(s_7)$$
$$V_\pi(s_5) = 0 + V_\pi(s_8)$$
$$V_\pi(s_6) = 1 + V_\pi(s_9)$$
$$V_\pi(s_7) = 0 + V_\pi(s_8)$$
$$V_\pi(s_8) = 1 + V_\pi(s_9)$$
$$V_\pi(s_9) = 0$$

Let $V_\pi = \left(V_\pi(s_1), \ldots, V_\pi(s_9)\right)^\top$, we have the following matrix form of the equations:

$$V_\pi = AV_\pi + b$$

In which $A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

This is a type of fixed-point equation

$x = f(x)$ in which $f(x) = \mathrm{A}x + \mathrm{b}$

that can be solved by fixed-point iteration as follows:

```
Randomly initialize x
while |x − f(x)| > ϵ:
      x ← f(x)
```

Its convergence can be proved with contraction mapping theorem

Policy Improvement

π

↓	↓	↓
↓	↓	↓
→	→	o

V_π

0	1	0
0	1	1
1	1	0

π'

→	↓	←
→	↓	↓
→	→	o

Policy improvement: update $\pi(s)$ for a larger $V_\pi(s)$

Recall the Bellman equation

$$V_\pi(s) = r\big(s, \pi(s)\big) + V_\pi(s') = r\big(s, \pi(s)\big) + V_\pi(P(s, \pi(s)))$$

How should we update $\pi(s)$ to enlarge $V_\pi(s)$?

$\pi(s_1)$	$V_\pi(s_1) = r\big(s_1, \pi(s_1)\big) + V_\pi(P(s_1, \pi(s_1)))$
$\boldsymbol{a_1}$ (↑)	$r(s_1, a_1) + V_\pi(P(s_1, a_1)) = -1 + V_\pi(s_1) = -1$
$\boldsymbol{a_2}$ (→)	$r(s_1, a_2) + V_\pi(P(s_1, a_2)) = 0 + V_\pi(s_2) = \mathbf{1}$
$\boldsymbol{a_3}$ (↓)	$r(s_1, a_3) + V_\pi(P(s_1, a_3)) = 0 + V_\pi(s_4) = 0$
$\boldsymbol{a_4}$ (←)	$r(s_1, a_4) + V_\pi(P(s_1, a_4)) = -1 + V_\pi(s_1) = -1$
$\boldsymbol{a_5}$ (o)	$r(s_1, a_5) + V_\pi(P(s_1, a_5)) = 0 + V_\pi(s_1) = 0$

The largest!

Let

$$q_\pi(s, a) = r(s, a) + V_\pi(P(s, a))$$

Then we update $\pi(s)$ by

$$\pi'(s) = \arg\max_a q_\pi(s, a)$$

for each $s \in S$

For the grid world example, we improve

- $\pi'(s_1) \leftarrow \arg\max_a q_\pi(s_1, a) = a_2$
- $\pi'(s_3) \leftarrow \arg\max_a q_\pi(s_3, a) = a_4$
- $\pi'(s_4) \leftarrow \arg\max_a q_\pi(s_4, a) = a_2$

(marked in red)

The policy π is optimal when we cannot do any further improvement. I.e.,

$$\pi(s) = \arg\max_a q_\pi(s, a) \text{ for all } s \in S$$

Policy Iteration

We iteratively do policy evaluation and improvement, until no further improvement can be done.

```
Randomly initialize π(s) and V_π(s)
While True:
    While |V_π(s) − (R + V_π(s′))| > ϵ:
        V_π(s) ← R + V_π(s′) for all s ∈ S
    π′(s) ← argmax_a q_π(s,a) for all s ∈ S
    if π′(s) = π(s): break
    π(s) ← π′(s)
```

Policy evaluation: compute $V_\pi(s)$ for the current policy $\pi(s)$ by fixed-point iteration

Policy improvement: improve $\pi(s)$ by selecting the action a with maximal $q(s,a)$

$R = r\big(s, \pi(s)\big)$ is the immediate reward
$s' = P(s, \pi(s))$ is the next state
$q_\pi(s,a) = r(s,a) + V_\pi(P(s,a))$

Value Iteration

Recall that the policy π is optimal when we cannot do any further improvement. I.e.,

$$\pi^*(s) = \arg\max_a q(s,a) \text{ for all } s \in S$$

So for an optimal policy π^*, its value function

$$V_{\pi^*}(\mathrm{s}) = q_{\pi^*}(s, \pi(s))$$

will always equal to the maximal one among $q(s,a)$. I.e.,

$$V_{\pi^*}(\mathrm{s}) = \max_a \; q_\pi(s,a)$$

$$V_{\pi^*}(\mathrm{s}) = \max_a \left(r(s,a) + V_{\pi^*}(P(s,a))\right)$$

This is also a fixed-point equation that can be solved by fixed-point iteration.

Randomly initialize $V_\pi(s)$

While $\left|V_\pi(s) - \max_a \left(r(s,a) + V_\pi(P(s,a))\right)\right| > \epsilon$:

$\quad V_\pi(s) \leftarrow \max_a \left(r(s,a) + V_\pi(P(s,a))\right)$ for all $s \in S$

$\pi(s) \leftarrow \arg\max_a q_\pi(s,a)$ for all $s \in S$

Its convergence can be proved with contraction mapping theorem

Value Iteration

s_1	s_2	s_3
s_4	s_5	s_6 (Trap)
s_7 (Trap)	s_8	s_9 (Target)

State \ Action	a_1 (↑)	a_2 (→)	a_3 (↓)	a_4 (←)	a_5 (o)
s_1	-1	0	0	-1	0
s_2	-1	0	0	0	0
s_3	-1	-1	-1	0	0
s_4	0	0	-1	-1	0
s_5	0	-1	0	0	0
s_6	0	-1	+1	0	-1
s_7	0	0	-1	-1	-1
s_8	0	+1	-1	-1	0
s_9	-1	-1	-1	0	+1

Example:

$$V_{\pi^*}(s) = \max_{a \in \{a_1, \dots, a_5\}} \big(r(s,a) + V_{\pi^*}(P(s,a))\big)$$

$$V_{\pi^*}(s_1) = \max\big(-1 + V_{\pi^*}(s_1),\ 0 + V_{\pi^*}(s_2),\ 0 + V_{\pi^*}(s_4), -1 + V_{\pi^*}(s_1),\ 0 + V_{\pi^*}(s_1)\big)$$
$$V_{\pi^*}(s_2) = \max\big(-1 + V_{\pi^*}(s_2),\ 0 + V_{\pi^*}(s_3),\ 0 + V_{\pi^*}(s_5),\ 0 + V_{\pi^*}(s_1),\ 0 + V_{\pi^*}(s_2)\big)$$
$$V_{\pi^*}(s_3) = \max\big(-1 + V_{\pi^*}(s_3), -1 + V_{\pi^*}(s_3), -1 + V_{\pi^*}(s_6),\ 0 + V_{\pi^*}(s_2),\ 0 + V_{\pi^*}(s_3)\big)$$
$$V_{\pi^*}(s_4) = \max\big(\ 0 + V_{\pi^*}(s_1),\ 0 + V_{\pi^*}(s_5), -1 + V_{\pi^*}(s_7), -1 + V_{\pi^*}(s_4),\ 0 + V_{\pi^*}(s_4)\big)$$
$$V_{\pi^*}(s_5) = \max\big(\ 0 + V_{\pi^*}(s_2), -1 + V_{\pi^*}(s_6),\ 0 + V_{\pi^*}(s_8),\ 0 + V_{\pi^*}(s_4),\ 0 + V_{\pi^*}(s_5)\big)$$
$$V_{\pi^*}(s_6) = \max\big(\ 0 + V_{\pi^*}(s_3), -1 + V_{\pi^*}(s_6), +1 + V_{\pi^*}(s_9),\ 0 + V_{\pi^*}(s_3), -1 + V_{\pi^*}(s_6)\big)$$
$$V_{\pi^*}(s_7) = \max\big(\ 0 + V_{\pi^*}(s_4), -1 + V_{\pi^*}(s_8), -1 + V_{\pi^*}(s_7),\ 0 + V_{\pi^*}(s_7), -1 + V_{\pi^*}(s_7)\big)$$
$$V_{\pi^*}(s_8) = \max\big(\ 0 + V_{\pi^*}(s_5), +1 + V_{\pi^*}(s_9), -1 + V_{\pi^*}(s_8), -1 + V_{\pi^*}(s_2),\ 0 + V_{\pi^*}(s_8)\big)$$
$$V_{\pi^*}(s_9) = 0$$

Let $V_{\pi^*} = \big(V_{\pi^*}(s_1), \dots, V_{\pi^*}(s_9)\big)^{\top}$, we have

$$V_{\pi^*} = f(V_{\pi^*})$$

in which $f(V_{\pi^*}) = \max(A_1 V_{\pi^*} + b_1, A_2 V_{\pi^*} + b_2, A_3 V_{\pi^*} + b_3, A_4 V_{\pi^*} + b_4, A_5 V_{\pi^*} + b_5)$

Its convergence can be proved with contraction mapping theorem

Stochastic Cases

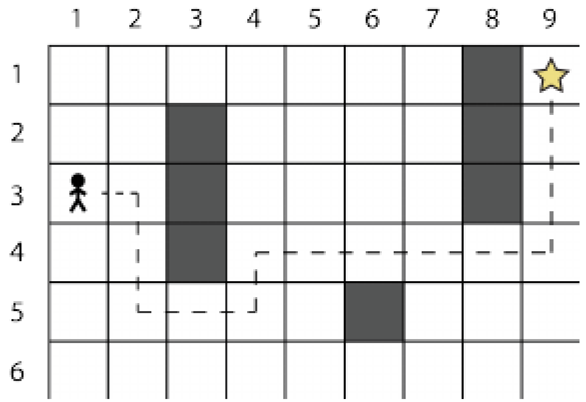
	Deterministic	Stochastic
State Transition	$P(s,a) = s'$	$P(s'\|s,a)$ is the conditional probability of transition from (s,a) to s'. $\sum_{s'\in S} P(s'\|s,a) = 1$
Agent's Policy	$\pi(s) = a$	$\pi(a\|s)$ is the conditional probability from s to a, $\sum_{a\in A} \pi(a\|s) = 1$
Return (Value)	$V_\pi(s) = R_1 + R_2 + \cdots$	$V_\pi(s) = \mathbb{E}[R_1 + R_2 + \cdots]$ (the expectation of total rewards)
Bellman Equation	$V_\pi(s) = r\big(s,\pi(s)\big) + V_\pi(s')$ in which $s' = P(s,\pi(s))$	$V_\pi(s) = \sum_{a\in A} \pi(a\|s)[r(s,a) + \sum_{s'\in S} P(s'\|s,a)\, V_\pi(s')]$
Action-Value Function	$q(s,a) = r(s,a) + V_\pi(P(s,a))$ $V_\pi(s) = q(s,\pi(s))$	$q(s,a) = r(s,a) + \sum_{s'\in S} P(s'\|s,a)\, V_\pi(s')$ $V_\pi(s) = \sum_{a\in A} \pi(a\|s) q(s,a)$

Decay factor γ

- Future rewards should be "less important"
- Sometimes critical for convergence
- γ is typically a number a bit less than 1 (e.g., 0.99)

	Without decay factor	With decay factor
Return (Value)	$V_\pi(s) = \mathbb{E}[R_1 + R_2 + R_3 + \cdots]$	$V_\pi(s) = \mathbb{E}[R_1 + \boldsymbol{\gamma} R_2 + \boldsymbol{\gamma}^2 R_3 + \cdots]$
Bellman Equation	$V_\pi(s) = \sum_{a \in A} \pi(a\|s)[r(s,a) + \sum_{s' \in S} P(s'\|s,a) V_\pi(s')]$	$V_\pi(s) = \sum_{a \in A} \pi(a\|s)[r(s,a) + \boldsymbol{\gamma} \sum_{s' \in S} P(s'\|s,a) V_\pi(s')]$
Action-Value Function	$q(s,a) = r(s,a) + \sum_{s' \in S} P(s'\|s,a) V_\pi(s')$	$q(s,a) = r(s,a) + \boldsymbol{\gamma} \sum_{s' \in S} P(s'\|s,a) V_\pi(s')$

Model-Based and Model-Free Setting



In a **model-based** setting, you have full access to the state transition $P(s'|s,a)$ and reward function $r(s,a)$

Model = state transition + reward function

Game developer's view

In a **model-free** setting, you don't have access to $P(s'|s,a)$ and $r(s,a)$, but can interact with the environment in an episodic way

$$s \xrightarrow[R]{a=\pi(s)} s' \xrightarrow[R']{a'=\pi(s')} s'' \xrightarrow[R'']{a''=\pi(s'')} \ldots$$

Player's view

For most complex, real-world settings

Learning in Model-Free Setting

Option 1: Learn a model (reduce the problem to model-based learning)
Estimate the state transition $P(s'|s,a)$ and reward function $r(s,a)$ of the given environment by episodic interaction.

Representative work: Dyna-Q
More detailed introduction can be found in
https://snowkylin.github.io/blogs/introduction-to-model-based-rl.html

A mathematician is pursuing a work as a firefighter.

"What do you do if you pass a Dumpster, and it's on fire?"

"Easy, I'd just put out the fire."

"Okay, what do you do if you pass a Dumpster, and it's not on fire?"

"Easy! I'd set it on fire!"

“Are you an idiot?!”

"No! I've just reduced the problem to one I've already solved!"

Learning in Model-Free Setting

Option 2: Learn a policy without a model

Improve agent's policy π

- without explicit involvement of $P(s'|s,a)$ (we don't know the exact probability)
- only involve $r(s,a)$ in an episodic setting. I.e., the agent is at state s and actually performed action a

Is it possible? Let's review the policy iteration method

Randomly initialize $\pi(s)$ and $V_\pi(s)$

While True:

> While $|V_\pi(s) - \left(R + V_\pi(s')\right)| > \epsilon$:
>
> $\qquad V_\pi(s) \leftarrow R + V_\pi(s')$ for all $s \in S$
>
> $\pi'(s) \leftarrow \arg\max_a q_\pi(s,a)$ for all $s \in S$
>
> if $\pi'(s) = \pi(s)$: break
>
> $\pi(s) \leftarrow \pi'(s)$

Policy evaluation: compute $V_\pi(s)$ for the current policy $\pi(s)$

Policy improvement: improve $\pi(s)$ by selecting the action a with maximal $q(s,a)$

$R = r\left(s, \pi(s)\right)$ is the immediate reward

$s' = P(s, \pi(s))$ is the next state

$q_\pi(s,a) = r(s,a) + V_\pi(P(s,a))$

Learning in Model-Free Setting

Option 2: Learn a policy without a model

Can we solve the problem if we have a way to estimate $V_\pi(s)$ without a model? Better not.
In policy improvement step, we need to compute

$$q_\pi(s,a) = r(s,a) + V_\pi(P(s,a))$$

Which require us to know the state transition function.

```
Randomly initialize π(s) and V_π(s)
While True:
    Estimate V_π(s) by episodic interaction
    with the environment using π(s)
    π'(s) ← arg max_a q_π(s,a) for all s ∈ S
    if π'(s) = π(s): break
    π(s) ← π'(s)
```

Policy evaluation: estimate $V_\pi(s)$ for the current policy $\pi(s)$

Policy improvement: improve $\pi(s)$ by selecting the action a with maximal $q(s,a)$

$$q_\pi(s,a) = r(s,a) + V_\pi(P(s,a))$$

Learning in Model-Free Setting

Option 2: Learn a policy without a model

How about estimate $q_\pi(s,a)$ directly instead of $V_\pi(s)$?

It works!

Randomly initialize $\pi(s)$ and $V_\pi(s)$

While True:

Estimate $q_\pi(s,a)$ by episodic interaction with the environment using $\pi(s)$

$\pi'(s) \leftarrow \arg\max_a q_\pi(s,a)$ for all $s \in S$

if $\pi'(s) = \pi(s)$: break

$\pi(s) \leftarrow \pi'(s)$

Policy evaluation: estimate $q_\pi(s,a)$ for the current policy $\pi(s)$

Policy improvement: improve $\pi(s)$ by selecting the action a with maximal $q(s,a)$

$$q_\pi(s,a) = r(s,a) + V_\pi(P(s,a))$$

Policy Evaluation - Estimating $\boldsymbol{q_{\pi}(s, a)}$ in an episodic setting

- $q_{\pi}(s, a)$ is the expectation of total rewards when starts from state s and performs action a.
- We can interact with the environment many times, to obtain lots of trajectories
- Then we do some simple statistics on the sampled trajectories:

$q_{\pi}(s, a) \approx$ the average return of (partial) trajectories
starting from s with action a

- Monte Carlo approach

Policy Evaluation - Estimating $q_\pi(s,a)$ with Temporal Difference Approach

- We can do incremental update on $q_\pi(s,a)$, even if the episode does not finish
- Assume we have a partial trajectory …, $s \xrightarrow[r]{a} s' \xrightarrow[r']{a'}, \ldots$
- We update $q_\pi(s,a)$ by

$$q_\pi(s,a) \leftarrow q_\pi(s,a) + \alpha\Big(\underbrace{r + q_\pi(s',a')}_{\text{New estimation based on the trajectory}} - \underbrace{q_\pi(s,a)}_{\text{Previous estimation}}\Big)$$

A small “update rate”, e.g., 0.01

Policy Improvement - Including Exploration in Agent's Policy

- We are estimating $Q_\pi(s, a)$ instead of precisely computing it, which means it can be inaccurate
- If we completely trust the estimated result to improve our policy, it may fall into suboptimality
  - Some (s, a) pair may never occur during the estimation
- We want to guarantee that every (s, a) has a positive probability during estimation.
- ϵ-greedy approach:
  - With probability 1-ε, choose the greedy action
  - With probability ε, choose an action at random

Left:
20% Reward = 0
80% Reward = 5

Right:
50% Reward = 1
50% Reward = 3

What if the first action is to choose the left door and observe reward=0?

SARSA Algorithm

Sarsa: An on-policy TD control algorithm

Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Choose A from S using policy derived from Q (e.g., ϵ-greedy)
  Repeat (for each step of episode):
    Take action A, observe R, S'
    Choose A' from S' using $\boxed{\text{policy derived from } Q \text{ (e.g., } \epsilon\text{-greedy)}}$
    $\boxed{Q(S,A) \leftarrow Q(S,A) + \alpha\big[R + \gamma Q(S',A') - Q(S,A)\big]}$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until S is terminal

Policy improvement with exploration

Policy evaluation with temporal difference

Q-Learning

Q-learning (off-policy TD control) for estimating $\pi \approx \pi_*$

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
Initialize $Q(s, a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(terminal, \cdot) = 0$

Loop for each episode:
  Initialize S
  Loop for each step of episode:
    Choose A from S using $\boxed{\text{policy derived from } Q \text{ (e.g., } \varepsilon\text{-greedy)}}$
    Take action A, observe R, S'
    $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \boxed{\gamma \max_a Q(S', a)} - Q(S, A)\big]$
    $S \leftarrow S'$
  until S is terminal

Similar to Value Iteration: not only estimate q_π but also optimize q_π towards optimal q_{π^*}

Off-policy: the policy that interacts with the environment and the policy in TD update can be different

Policy Gradient

- Assume the agent's policy $\pi(a|s)$ is parameterized by θ (denoted by π_θ) and differentiable
- Our target is to find an optimal policy π_θ^* that maximize the (expected) return $V_{\pi_\theta}(s)$
- Can we achieve this by gradient descend (ascend)?
- Let the objective be

$$J(\theta) = \mathbb{E}_{s_0}[V_{\pi_\theta}(s_0)]$$

in which s_0 is the initial state

- The gradient is

$$\frac{\partial}{\partial\theta}J(\theta) = \mathbb{E}_{\pi_\theta}[q_{\pi_\theta}(s,a)\frac{\partial}{\partial\theta}\log\pi_\theta(a|s)]$$

Thank you!

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