Matrix Game, Markov Game, POMDP, PSR

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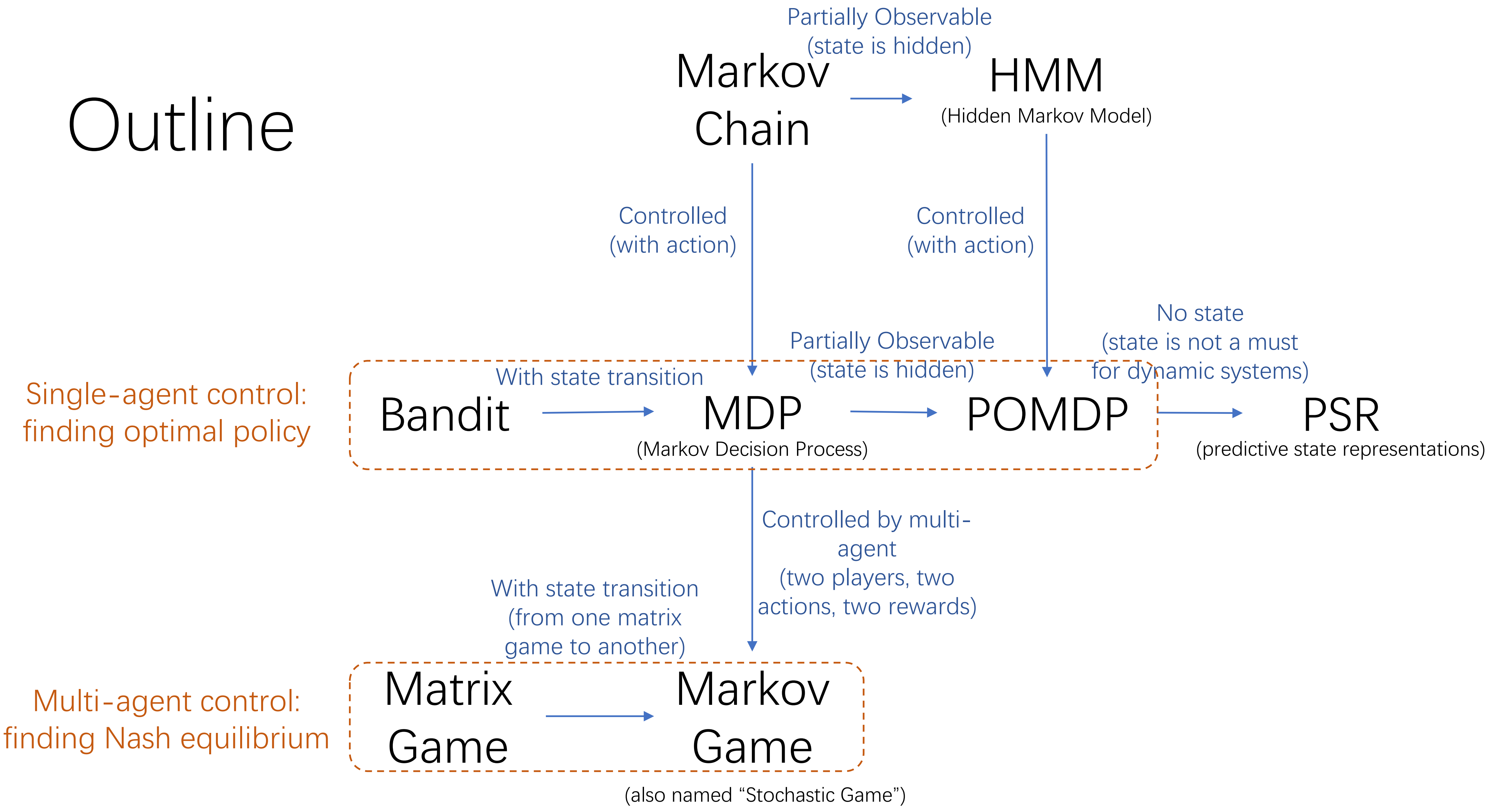
Contents based on https://sites.google.com/view/cjin/ele524

Outline



Partially Observable
(state is hidden)
Markov Chain
HMM
(Hidden Markov Model)
Controlled
(with action)
Controlled
(with action)
No state
(state is not a must
for dynamic systems)
Partially Observable
(state is hidden)
With state transition
Single-agent control:
finding optimal policy
Bandit
MDP
(Markov Decision Process)
POMDP
PSR
(predictive state representations)
Controlled by multi-
agent
(two players, two
actions, two rewards)
With state transition
(from one matrix
game to another)
Multi-agent control:
finding Nash equilibrium
Matrix Game
Markov Game
(also named "Stochastic Game")

Matrix Game

- A set of players
  - e.g., you (row player, player 1) and your opponent (column player, player 2)
- Each player chooses an action
  - e.g., $\mathcal{A} = \mathcal{B} = \{rock, paper, scissor\}$, you choose $a \in \mathcal{A}$, your opponent choose $b \in \mathcal{B}$
- Each player receives a reward
  - e.g., when you choose $a = rock$ and your opponent choose $b = paper$, you receive reward -1 (lose) and your opponent receive 1 (win)
  - More generally, when you choose a and your opponent choose b, you receive $R_1(a, b)$ and your opponent receive $R_2(a, b)$
- Zero-sum game: $R_1(a, b) + R_2(a, b) + \cdots = 0$
  - So we can use a single function $R(a, b)$ to denote the reward in 2-player setting

Matrix Game: policy (strategy)

Your opponent's action b

Your action a

$R_1(a,b)$	Rock	Paper	Scissor
Rock	0	-1	1
Paper	1	0	-1
Scissor	-1	1	0

Your reward

Your opponent's action b

Your action a

$R_2(a,b)$	Rock	Paper	Scissor
Rock	0	1	-1
Paper	-1	0	1
Scissor	1	-1	0

Your opponent's reward

Here we have action $a \in \mathcal{A}$, what about the policy $\pi(\cdot) \in \Delta_{\mathcal{A}}$?
Different from MDP, we don't have state here.

- MDP: deterministic policy $\pi: \mathcal{S} \to \mathcal{A}$ or stochastic policy $\pi: \mathcal{S} \to \Delta_{\mathcal{A}}$
- Matrix Game: **pure strategy** $\mu \in \mathcal{A}, \nu \in \mathcal{B}$ or **mixed strategy** $\mu \in \Delta_{\mathcal{A}}, \nu \in \Delta_{\mathcal{B}}$

$\Delta_{\mathcal{A}}$: Distribution (simplex) over action set $\mathcal{A}$. E.g., $(0.3, 0.3, 0.4) \in \Delta_{\mathcal{A}}$, which means you have probability 0.3 to play rock or paper, probability 0.4 to play scissor.

E.g.

- Pure strategy: $\mu = rock$ (you always play rock), $\nu = paper$ (your opponent always play paper), $R(\mu, \nu) = -1$ (you always lose)
- Mixed strategy: $\mu = (\frac{1}{2}, \frac{1}{2}, 0)$ (you play $\frac{1}{2}$ rock, $\frac{1}{2}$ paper), $\nu = (0, \frac{1}{2}, \frac{1}{2})$ (your opponent play $\frac{1}{2}$ paper, $\frac{1}{2}$ scissor)

Matrix Game: reward

Your opponent's action b

Your action a

$R_1(a,b)$	Rock	Paper	Scissor
Rock	0	-1	1
Paper	1	0	-1
Scissor	-1	1	0

Your reward

Your opponent's action b

Your action a

$R_2(a,b)$	Rock	Paper	Scissor
Rock	0	1	-1
Paper	-1	0	1
Scissor	1	-1	0

Your opponent's reward

$$[0.5 \quad 0.5 \quad 0] \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} = -0.25$$

What about the reward $\mathrm{r} \in \mathbb{R}$?

Different from MDP, we have separate rewards for each player.

Beside your action, your reward is also determined by what your opponent plays.

- MDP: $r(s,a)$: $\mathcal{S} \times \mathcal{A} \to \mathbb{R}$
- Matrix Game: $R_1(a,b), R_2(a,b)$: $\mathcal{A} \times \mathcal{B} \to \mathbb{R}$ (matrices $R_i \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{B}|}$)

Expected reward for player i: $f_i(\mu,\nu) = \mathbb{E}_{a\sim\mu, b\sim\nu}[R_i(\mu,\nu)] = \sum_{a,b} \mu(a) R_i(\mu,\nu)\nu(b) = \mu^\top R_i \nu$

E.g. Mixed strategy: $\mu = (\frac{1}{2}, \frac{1}{2}, 0)$ (you play $\frac{1}{2}$ rock, $\frac{1}{2}$ paper), $\nu = (0, \frac{1}{2}, \frac{1}{2})$ (your opponent play $\frac{1}{2}$ paper, $\frac{1}{2}$ scissor)

Your expected reward: $\mathbb{E}_{a\sim\mu, b\sim\nu}[R(\mu,\nu)] = \frac{1}{4} \times (-1) + \frac{1}{4} \times 1 + \frac{1}{4} \times 0 + \frac{1}{4} \times (-1) = -\frac{1}{4}$

Matrix Game: best response

Your opponent's action b

Your action a

$R_1(a,b)$	Rock	Paper	Scissor
Rock	0	-1	1
Paper	1	0	-1
Scissor	-1	1	0

Your reward

Your opponent's action b

Your action a

$R_2(a,b)$	Rock	Paper	Scissor
Rock	0	1	-1
Paper	-1	0	1
Scissor	1	-1	0

Your opponent's reward

What about the reward and the optimal policy π^*?
In matrix game, our optimal policy (strategy) is relevant to the policy (strategy) of the opponent.

- MDP: optimal policy $\pi^* = \arg\max_{\pi} \mathbb{E}_{a\sim\pi}[\sum r_t]$ (the policy that maximize the cumulated reward)
- Matrix Game: **best response for you** $\mu^*(\nu) = \arg\max_{\mu} \mathbb{E}_{a\sim\mu, b\sim\nu}[R_1(a,b)]$ (the strategy that maximize the reward given ν), **best response for your opponent** $\nu^*(\mu) = \arg\max_{\nu} \mathbb{E}_{a\sim\mu, b\sim\nu}[R_2(a,b)]$

E.g.

- When your opponent play $\nu = (1,0,0)$ (always play rock), your best response is $v = (0,1,0)$ (always play paper) so that $\mathbb{E}_{a\sim\mu, b\sim\nu}[R(a,b)] = 1$ is maximized.

Matrix Game: Nash equilibrium

Your opponent's action b

Your action a

$R_1(a,b)$	Rock	Paper	Scissor
Rock	0	-1	1
Paper	1	0	-1
Scissor	-1	1	0

Your reward

Your opponent's action b

Your action a

$R_2(a,b)$	Rock	Paper	Scissor
Rock	0	1	-1
Paper	-1	0	1
Scissor	1	-1	0

Your opponent's reward

Is there some "optimal policy (strategy)" that does not depend on the opponent's policy (strategy)?

A **Nash Equilibrium** is a strategy (μ, ν) such that neither player will gain anything by deviating from his own strategy while the opposing player continues to play its current strategy.
E.g., $\mu = \nu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a Nash equilibrium (you cannot increase your expectation of reward if your opponent plays rock, paper and scissor equality, vice versa.)

Theorem: *Every game with a finite number of players and action profiles has at least one Nash equilibrium.*
Complete Proof: https://www.cs.ubc.ca/~jiang/papers/NashReport.pdf

Zero-sum, 2-player Nash equilibrium proof

- Let

$$f(\mu,\nu) = \mathbb{E}_{a\sim\mu,b\sim\nu}[R(\mu,\nu)] = \sum_{a,b}\mu(a)R(\mu,\nu)\nu(b) = \mu^\top R\nu$$

- be your expected reward ($-f(\mu,\nu)$ for your opponent)
- In zero-sum, 2-player setting, a strategy pair (μ^*,ν^*) is a Nash equilibrium if
  - Your expected reward $f(\mu^*,\nu^*) \geq \max_\mu f(\mu,\nu^*)$
  - Your opponent's expected reward $-f(\mu^*,\nu^*) \geq \max_\nu -f(\mu^*,\nu) \Rightarrow f(\mu^*,\nu^*) \leq \min_\nu f(\mu^*,\nu)$
- That is,

$$\boxed{\max_x -f(x) = -\min_x f(x)}$$

$$\max_\mu f(\mu,\nu^*) \leq f(\mu^*,\nu^*) \leq \min_\nu f(\mu^*,\nu) \ \forall \mu,\nu \in \Delta_{\mathcal{A}}$$

Which means

$$\min_\nu \max_\mu f(\mu,\nu) \leq f(\mu^*,\nu^*) \leq \max_\mu \min_\nu f(\mu,\nu)$$

$$\boxed{\min_\nu[\max_\mu f(\mu,\nu)] \leq \max_\mu f(\mu,\nu^*)}$$

Zero-sum, 2-player Nash equilibrium proof

Lemma:

$$\min_b \max_a f(a,b) \geq \max_a \min_b f(a,b)$$

Proof:

$$f(a,b) \geq \min_b f(a,b) \ \ \forall a,b$$

$$\max_a f(a,b) \geq \max_a \min_b f(a,b) \ \ \forall b$$

$$\min_b \max_a f(a,b) \geq \max_a \min_b f(a,b)$$

$\blacksquare$

Ref: Minimax theorem, game theory and Lagrange duality https://www.youtube.com/watch?v=MFEkxYuoFqw

Zero-sum, 2-player Nash equilibrium proof

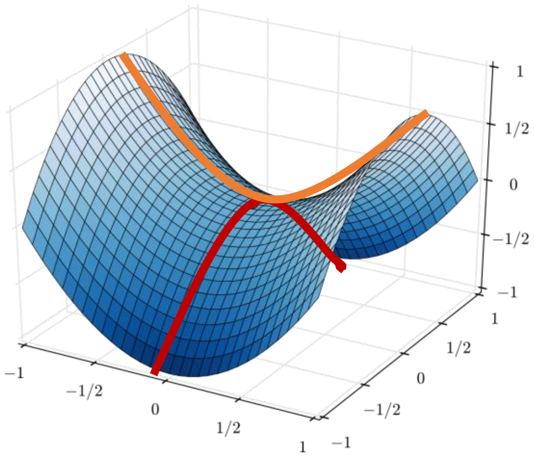
So, the existence of a Nash Equilibrium implies that

$$\min_{\nu}\max_{\mu} f(\mu,\nu) = f(\mu^*,\nu^*) = \max_{\mu}\min_{\nu} f(\mu,\nu)$$

Von Naumann's minimax theorem: Let $\mathcal{U},\mathcal{V}$ be convex, compact sets, $f: \mathcal{U}\times\mathcal{V}\to\mathbb{R}$ is a convex-concave continuous function (meaning that $f(\mu,\cdot)$ is convex $\forall\mu$ and $f(\cdot,\nu)$ is concave $\forall\nu$). Then

$$\min_{\nu}\max_{\mu} f(\mu,\nu) = \max_{\mu}\min_{\nu} f(\mu,\nu)$$

■



In our case, $f(\mu,\nu) = \mu^{\top}R\nu$ is bi-linear (convex-concave), $\mu\in\Delta_{\mathcal{A}},\nu\in\Delta_{\mathcal{B}}$ are simplex (convex)

Both the left and right side are constrained optimization problems

$$\min_{\nu}\max_{\mu}\mu^{\top}R\nu \;\; s.t.\; \mu\in\Delta_{\mathcal{A}},\nu\in\Delta_{\mathcal{B}}$$

$$\max_{\mu}\min_{\nu}\mu^{\top}R\nu \;\; s.t.\; \mu\in\Delta_{\mathcal{A}},\nu\in\Delta_{\mathcal{B}}$$

Which can be transformed as Linear Programming models. By the duality of Linear Programming, the equality also holds.

Finding Nash equilibrium

- (projected) gradient descent ascent

$$x_{t+1} = x_t + \eta \partial_x f(x_t, y_t)$$
$$y_{t+1} = y_t - \eta \partial_y f(x_t, y_t)$$

Markov Games

Matrix Game with state and transitions.

- Each state $s \in \mathcal{S}$ is a Matrix game.
- $P(s'|s, a, b)$ is the transition probability (with multiple actions)
- $r_i(s, a, b)$ is the reward of player $i = 1,2$ for Matrix Game s when player 1 plays a and player 2 plays b.

Policy:

- MDP: deterministic policy $\pi: \mathcal{S} \rightarrow \mathcal{A}$ or stochastic policy $\pi: \mathcal{S} \rightarrow \Delta_{\mathcal{A}}$
- Matrix Game: pure strategy $\mu \in \mathcal{A}, \nu \in \mathcal{B}$ or mixed strategy $\mu \in \Delta_{\mathcal{A}}, \nu \in \Delta_{\mathcal{B}}$
- Markov Game: deterministic policy $\mu: \mathcal{S} \rightarrow \mathcal{A}, \nu: \mathcal{S} \rightarrow \mathcal{B}$ or stochastic policy $\mu: \mathcal{S} \rightarrow \Delta_{\mathcal{A}}, \nu: \mathcal{S} \rightarrow \Delta_{\mathcal{B}}$
  - very similar to MDP, but have multiple policies for multiple players

Markov Games

Lecture 3: MDP

Value function and expected reward for player i given a state s

$$V_i^{\mu,\nu}(s) = \mathbb{E}_{\mu,\nu}\left[G_i^t \middle| s_t = s\right], G_i^t = \sum_{k=t}^{T} r_i(s,a,b)$$

$$V_h^\pi(s) = \mathbb{E}_\pi\left[\sum_{h'=h}^{H} r_{h'}(s_{h'}, a_{h'}) | s_h = s\right]$$

State-action value function for player i

$$Q_i^{\mu,\nu}(s,a,b) = \mathbb{E}_{\mu,\nu}\left[G_i^t \middle| s_t = s, a_t = a, b_t = b\right]$$

$$Q_h^\pi(s,a) = \mathbb{E}_\pi\left[\sum_{h'=h}^{H} r_{h'}(s_{h'}, a_{h'}) | s_h = s, a_h = a\right]$$

Bellman Equation for player i

$$Q_i^{\mu,\nu}(s,a,b) = r_i(s,a,b) + \mathbb{E}_{s'\sim P(\cdot|s,a,b)} V_i^{\mu,\nu}(s)$$

$$V_i^{\mu,\nu}(s) = \sum_{a\in\mathcal{A}, b\in\mathcal{B}} \mu(s,a) Q_i^{\mu,\nu}(s,a,b)\nu(s,b)$$

$$= \mu(s)^\top Q_i^{\mu,\nu}(s)\nu(s)$$

$$\begin{cases} V_h^\pi(s) & = \sum_{a\in\mathcal{A}} Q_h^\pi(s,a)\pi_h(a|s) \\ Q_h^\pi(s,a) & = r_h(s,a) + \mathbb{E}_{s'\sim\mathbb{P}_h(\cdot|s,a)} V_{h+1}^\pi(s') \end{cases}$$

Similarly, when we are in a zero-sum, 2-player setting, we can write $r(s,a,b)$ directly without specifying the player.

Markov Games: best response & Nash Equilibrium

- If the policy of your opponent ν is given, the Markov Game becomes an MDP with optimal policy $\mu^*(\nu) = \operatorname{argmax} V_1^{\mu,\nu}$, which is called the best response. Similarly, $\nu^*(\mu) = \operatorname{argmax}_{\nu} V_2^{\mu,\nu}$.
- For 2-player zero-sum game, $V_2^{\mu,\nu} = -V_1^{\mu,\nu}$ (so we just write $V_1^{\mu,\nu} = V^{\mu,\nu}$)
- If (μ^*, ν^*) is a Nash equilibrium, then

$$V^{\mu^*,\nu^*} \geq \max_{\mu} V^{\mu,\nu^*}, -V^{\mu^*,\nu^*} \geq \max_{\nu} -V^{\mu^*,\nu} \Rightarrow V^{\mu^*,\nu^*} \leq \min_{\nu} V^{\mu^*,\nu}$$

$$\max_{\mu} V^{\mu,\nu^*} \leq V^{\mu^*,\nu^*} \leq \min_{\nu} V^{\mu^*,\nu}$$

Like the case in Matrix Game, we have

$$\min_{\nu} \max_{\mu} V^{\mu,\nu} = V^{\mu^*,\nu^*} = \max_{\mu} \min_{\nu} V^{\mu,\nu}$$

Replace $f(\mu,\nu)$ in Matrix Game to cumulated reward $V^{\mu,\nu}$

Markov Game: finding Nash equilibrium

Lecture 3: MDP

Bellman Equation: $V(s) = \sum_{a \in \mathcal{A}} \pi(a|s)\, Q(s,a)$

For all state $s \in S$:

$$V(s) = \max_{\pi \in \Delta_{\mathcal{A}}} \sum_{a \in \mathcal{A}} \pi(a|s) Q(s,a),$$ in which $Q(s,a) = r(s,a) + \mathbb{E}_{s' \sim P(s'|s,a)} V(s')$

$$= \max_{a \in \mathcal{A}} Q(s,a),$$ since $\pi(s) \in \Delta_{\mathcal{A}}$, vector π will be a one-hot vector when (greedily) maximized

For a fixed opponent policy ν, the Markov Game becomes an MDP, and we can find the best response via value iteration above

Bellman Equation: $V^{\mu,\nu}(s) = \sum_{a \in \mathcal{A}, b \in \mathcal{B}} \mu(s,a) Q^{\mu,\nu}(s,a,b) \nu(s,b) = \sum_{a \in \mathcal{A}} \mu(s,a) \left[\sum_{b \in \mathcal{B}} Q^{\mu,\nu}(s,a,b) \nu(s,b)\right]$

For all state $s \in S$:

$$V^{\mu,\nu}(s) = \max_{\mu \in \Delta_{\mathcal{A}}} \sum_{a \in \mathcal{A}, b \in \mathcal{B}} \mu(s,a) Q^{\mu,\nu}(s,a,b) \nu(s,b),$$ in which $Q^{\mu,\nu}(s,a,b) = r(s,a,b) + \mathbb{E}_{s' \sim P(\cdot|s,a,b)} V^{\mu,\nu}(s)$

$$= \max_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} Q^{\mu,\nu}(s,a,b) \nu(s,b)$$

To find the Nash equilibrium, we use

For all state $s \in S$:

$$V^{\mu,\nu}(s) = \min_{\nu \in \Delta_{\mathcal{B}}} \max_{\mu \in \Delta_{\mathcal{A}}} \sum_{a \in \mathcal{A}, b \in \mathcal{B}} \mu(s,a) Q^{\mu,\nu}(s,a,b) \nu(s,b)$$

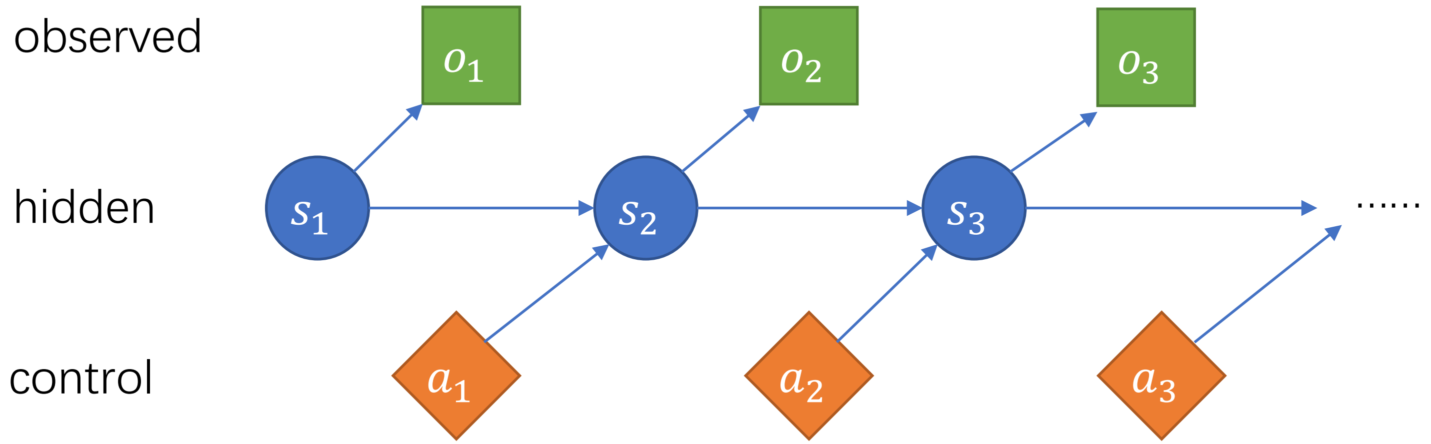
For each state $s \in \mathcal{S}$, finding a Nash Equilibrium for the Matrix Game with reward matrix $Q^{\mu,\nu}(s) \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{B}|}$

Partially Observable MDP

- The state of many applications are not fully observable
  - Poker (you don't know your opponent's hand)
  - StarCraft (fog)
- Need more general models to describe the problem

POMDP: adds observation $\mathcal{O}$ to the MDP formalization

- $\mathbb{O}(o|s)$: observation probability (under a state s, the possibility to observe o)
- $r(o)$: reward is a function of observation.



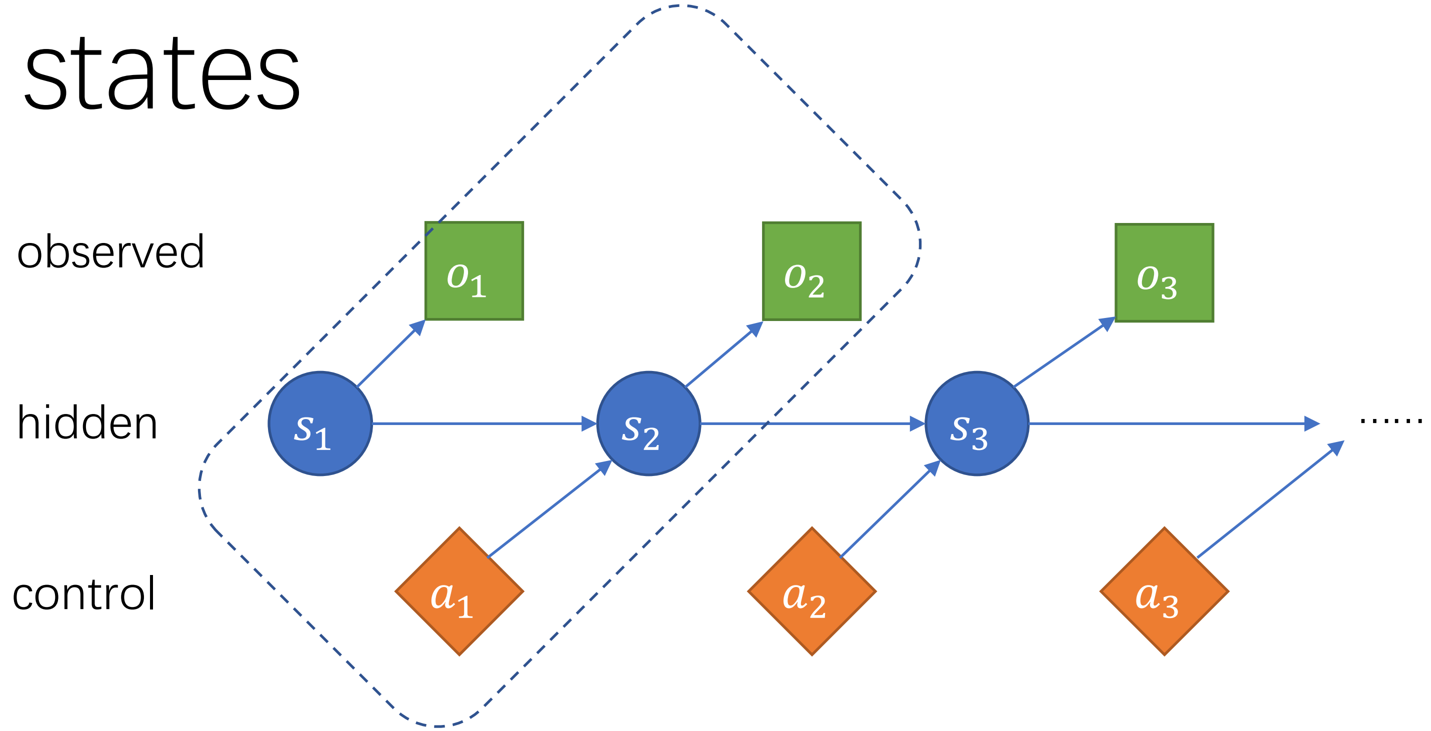
"History" and policy

- Instead of state s, decisions is based on the entire history

$$\tau_t = (o_1, a_1, o_2, a_2, \cdots, o_{t-1}, a_{t-1}, o_t)$$

- Policy is a mapping from history to (distribution of) action, $\pi(\tau) \in \Delta_{\mathcal{A}}$
- Bellman Equation
  - $V^\pi(\tau_t) = \sum_{a\in\mathcal{A}} \pi(a|\tau_t) Q^\pi(\tau_t, a)$
  - $Q^\pi(\tau_t, a_t) = \mathbb{E}_{o_{t+1}\sim P(\cdot|\tau_t, a_t)}[r(o_{t+1}) + V^\pi(\{\tau_t, a_t, o_{t+1}\})]$
- Bellman Optimality Equation
  - $V^*(\tau_t) = \max_{a\in\mathcal{A}} Q^*(\tau_t, a)$
  - $Q^*(\tau_t, a_t) = \mathbb{E}_{o_{t+1}\sim P(\cdot|\tau_t, a_t)}[r(o_{t+1}) + V^*(\{\tau_t, a_t, o_{t+1}\})]$
  - Optimal Policy $\pi^*(\tau_t) = \arg\max_{a\in\mathcal{A}} Q^*(\tau_t, a)$
- Planning in POMDP in general cannot be done computational efficiently.

Belief states



- History gives a distribution over s_2
  - If two histories generate the same belief states, then there should not be difference in the future. (i.e., earlier view has redundancy)
- Belief state $b_t \in \Delta_{\mathcal{S}}$ is a simplex (distribution) over all state $\mathcal{S}$
- Policy $\pi: \Delta_{\mathcal{S}} \to \Delta_{\mathcal{A}}$ only needs to rely on sufficient statistics
- POMDP $\Leftrightarrow$ belief-state MDP

Belief states

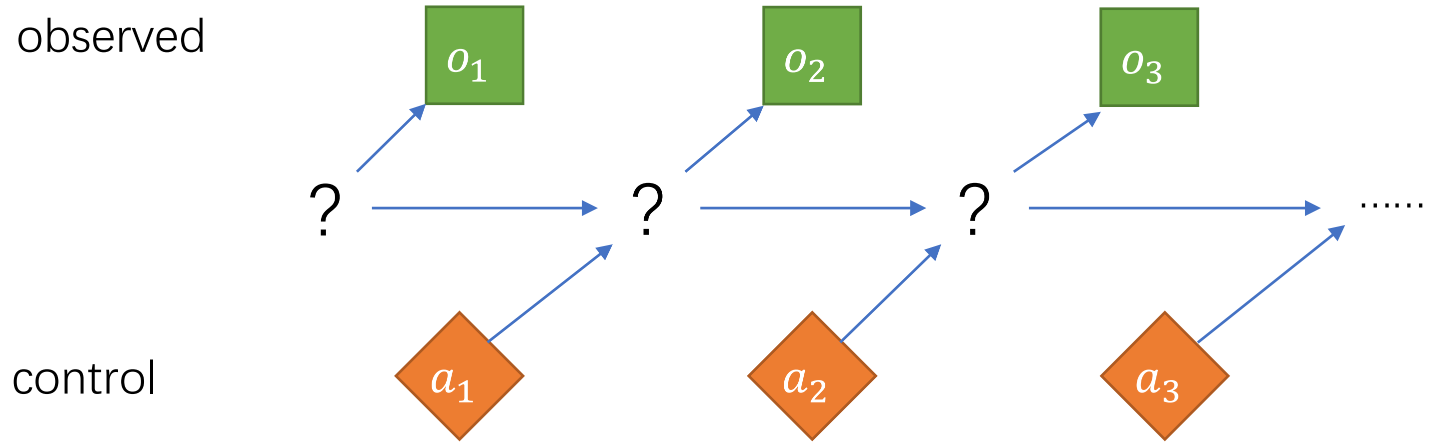
- Update on belief states: the probability of state $s_{t+1} \in \mathcal{S}$ in b_{t+1} is

$$b_{t+1}(s_{t+1}) = \frac{P(s_{t+1}, o_{t+1} | a_t, b_t)}{P(o_{t+1} | a_t, b_t)}$$

So b_{t+1} is a function of b_t, a_t, o_{t+1} $(b_{t+1} = f(b_t, a_t, o_{t+1}))$

- Bellman Equation
  - $V^\pi(b_t) = \sum_{a \in \mathcal{A}} \pi(a|b_t) Q^\pi(b_t, a)$
  - $Q^\pi(b_t, a_t) = \mathbb{E}_{o_{t+1} \sim P(\cdot|b_t, a_t)}[r(o_{t+1}) + V^\pi(f(b_t, a_t, o_{t+1}))]$
- In general, $V^*(b)$ is not a linear function in b
  - Still in general computationally intractable

Predictive State Representation



- State is not a must in dynamic systems
  - In practical applications, there may or may not exist interpretable hidden states. They may not be unique, nor "intrinsic"
- Define a test $t = (a^1, o^1, \cdots, a^k, o^k)$ of length k
- System-dynamics vector:

$$p(t) = \Pr(o_1 = o^1, \cdots, o_k = o^k | a_1 = a^1, \cdots, a_k = a^k)$$

- Once we know system dynamics vector, we know everything about the dynamic system

System-dynamic Matrix

$$p(t) = \Pr(o_1 = o^1, \cdots, o_k = o^k | a_1 = a^1, \cdots, a_k = a^k)$$

- It will be easier to see the structure in matrix form
- Test $t = (a^1, o^1, \cdots, a^k, o^k)$, history $h = (a_1, o_1, \cdots, a_l, o_l)$

$$p(t|h) = \Pr(o_{l+1} = o^1, \cdots, o_{l+k} = o^k | h, a_{l+1} = a^1, \cdots, a_{l+k} = a^k)$$

Concatenate h and t

$$P(t|h) = \frac{p(ht)}{p(h)}$$

Empty set Ø

	$t_0, \cdots\cdots,$	$t_i, \cdots\cdots$
h_0		
.		
.		
.		
h_j		$P(t_i|h_j)$
.		
.		
.		

System-dynamic matrix can be computed by system-dynamic vector

For POMDP with $|S|$ hidden states, $rank(SD\ matrix) \leq |S|$
Proof. $p(t|h) = \sum_s p(t|s)p(s|h) = b[h]^\top u_t$ (s-dimensional inner product)

Fact: There exists dynamic system whose $rank(SD\ matrix) = 3$, but cannot be represented by any finite POMDP

Core test Q and Predictive State Representation $\psi(h)$

- $Q = \{q_1, \cdots, q_k\}$, k columns of SD matrix, full column rank
- $\psi(h) = [p(q_1|h), \cdots, p(q_k|h)]$
- Then $p(t|h) = m_t^\top \psi(h)$
  - Predicting a new column t using core set.
  - Linear coefficient m_t should not depends on h
- $\psi(h)$ is called Predictive State Representation of h
  - A sufficient statistic, similar to the role of belief state